Math 2050, summary of Week 11

1. Continuous on closed and bounded interval, cont.

Recall that if the domain is closed and bounded, the behavior of the functions can't be too bad due to the constraints on the boundary (roughly speaking). In case $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the "boundary" is referring to the infinity ∞. Oppositely, if the behavior at ∞ is well-controlled, then the function is also well-controlled globally to some extent.

Example. If $p(x)$ is a polynomial of odd degree, then $p(x)$ has at least one real root.

Sketch of Proof in the lecture. Denote $p(x) = \sum_{i=0}^{2n+1} a_i x^i$. We may assume $a_{2n+1} > 0$, otherwise consider $-p(x)$. By the definition of limit, there is M such that for $|x| \geq M$,

$$\frac{p(x)}{a_{2n+1}x^{2n+1}} \geq \frac{1}{2}.$$

In particular, if $x \geq M$, then $p(x) > 0$ while if $x \leq -M$, $p(x) < 0$. The result follows from applying intermediate value theorem on $[-M, M]$. □

Example. If $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $\lim_{x\to+\infty} f(x) = \alpha$ and $\lim_{x\to-\infty} f(x) = \beta$ for some $\alpha, \beta \in \mathbb{R}$, then f is bounded on $\mathbb{R}$.

Sketch of Proof. By using definition of limit, there is M such that if $|x| \geq M$, we have

$$|f(x)| \leq |\alpha| + |\beta| + 1.$$

Then applying boundedness Theorem on $[-M, M]$, there is Λ such that for all $|x| \leq M$, $|f(x)| \leq \Lambda$. Combines two inequalities, we obtain an upper bound on $\mathbb{R}$.

□

2. Uniform continuity

Example. $f(x) = x^{-1}$ on $(0, 1)$. Recall that: To examine the continuity, we choose δ in the following way. For $c \in (0, 1)$,

$$|f(x) - f(c)| = \frac{1}{xc}|x - c|. \tag{2.1}$$

Hence, for $\varepsilon > 0$, we choose $\delta_c = \min\{\frac{c}{2}, \frac{1}{2}c^2\varepsilon\}$ so that for $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. In this way, δ_c depends on the choice of c

and more importantly, $\delta_c \to 0$ as $c \to 0$. This is the reason why we can't obtain uniform boundedness for f in this example!

To overcome this, we introduce a new concept.

Definition 2.1. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f is uniformly continuous on A if $\forall \varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

Clearly, an uniform continuous function is continuous. More importantly, the uniform continuity depends on the domain!

Theorem 2.1. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. The function f is NOT uniformly continuous if and only if $\exists \varepsilon_0 > 0$ and two sequence $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset A$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n.*

Example. $f(x) = x^{-1}$ is uniformly continuous on $(a, +\infty)$ if $a > 0$. And f is NOT uniformly continuous if $a = 0$.

Proof. We compute

$$|f(x) - f(c)| = \frac{1}{xc}|x - c| \leq a^{-2}|x - c|. \tag{2.2}$$

Hence, $\forall \varepsilon > 0$, $\exists \delta = a^2 \varepsilon$ so that if $|x - c| < \delta$ and $x, c > a$, we have $|f(x) - f(c)| < \varepsilon$.

If $a = 0$, the proof fails (but this is not a proof!). To show the non-uniform continuity, we choose $x_n = n^{-1}$, $y_n = 2n^{-1}$ so that $|x_n - y_n| \leq n^{-1} \to 0$ but

$$|f(x_n) - f(y_n)| = n - \frac{n}{2} = \frac{n}{2} \geq \frac{1}{2}. \tag{2.3}$$

□

Theorem 2.2. *Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is bounded.*

Proof. Let $\varepsilon = 1$, there is $\delta > 0$ such that if $|x - y| < \delta$ and $x, y \in (a, b)$, then

$$|f(x) - f(y)| < 1.$$

Let N be large enough so that $(b - a)/N < \delta$ and define $x_n = a + nN^{-1}(b - a)$ so that if $x \in (a, b)$, then $|x - x_n| < \delta$ for some $n = 1, 2, ..., N$. Hence,

$$\begin{aligned} |f(x)| &\leq |f(x) - f(x_n)| + |f(x_n)| \\ &\leq 1 + \max\{|f(x_i)| : i = 1, ..., N\} \\ &= M. \end{aligned} \tag{2.4}$$

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□